Integrability from an abelian subgroup of the diffeomorphism group

C. Adam[1a], J. Sánchez-Guillén[1b], and A. Wereszczyński[2c]

[1]Departamento de Física de Partículas,
Facultad de Física, Universidad de Santiago, and
Instituto Galego de Fisica de Altas Enerxias (IGFAE)
E-15782 Santiago de Compostela, Spain

[2]Institute of Physics, Jagiellonian University,
Reymonta 4, 30-059 Kraków, Poland

Abstract

It has been known for some time that for a large class of non-linear field theories in Minkowski space with two-dimensional target space the complex eikonal equation defines integrable submodels with infinitely many conservation laws. These conservation laws are related to the area-preserving diffeomorphisms on target space. Here we demonstrate that for all these theories there exists, in fact, a weaker integrability condition which again defines submodels with infinitely many conservation laws. These conservation laws will be related to an abelian subgroup of the group of area-preserving diffeomorphisms. As this weaker integrability condition is much easier to fulfil, it should be useful in the study of those non-linear field theories.

[a]adam@fpaxp1.usc.es
[b]joaquin@fpaxp1.usc.es
[c]wereszczynski@th.if.uj.edu.pl

1 Introduction

Recently there has been rising interest in non-linear field theories which allow for the existence of knotlike solitons. The probably best known of these models, the Faddeev–Niemi model [1, 2], for example, finds some applications in condensed matter physics [3, 4]. Further, some versions of it are discussed as possible candidates for a low-energy effective theory of Yang-Mills theory [5, 6]. In addition, there is some intrinsical mathematical interest in theories with knot solitons. Generally, these models are described by a complex field $u : \mathbb{R}^3 \times \mathbb{R} \to \mathcal{M} : (\vec{x}, t) \to u(\vec{x}, t)$ where $\mathcal{M}$ is a two-dimensional target space manifold and u plays the role of a complex coordinate on this manifold.

The Faddeev–Niemi model has the two-sphere as target space and is given by the Lagrangian density

$$\mathcal{L}_{\rm FN} = \mathcal{L}_2 - \lambda \mathcal{L}_4 \tag{1}$$

where λ is a dimensionful coupling constant, $\mathcal{L}_2$ is

$$\mathcal{L}_2 = 4\frac{\partial_\mu u\, \partial^\mu \bar{u}}{(1+u\bar{u})^2}, \tag{2}$$

and $\mathcal{L}_4$ is

$$\mathcal{L}_4 = 4\frac{(\partial^\mu u\, \partial_\mu \bar{u})^2 - (\partial^\mu u\, \partial_\mu u)(\partial^\nu \bar{u}\, \partial_\nu \bar{u})}{(1+u\bar{u})^4}. \tag{3}$$

Two more models which support solitons and can be constructed from the two Lagrangian densities $\mathcal{L}_2$ and $\mathcal{L}_4$ separately, are the AFZ (=Aratyn, Ferreira and Zimerman) model [7, 8]

$$\mathcal{L}_{\rm AFZ} = -(\mathcal{L}_4)^{\frac{3}{4}} \tag{4}$$

and the Nicole model [9]

$$\mathcal{L}_{\rm Ni} = (\mathcal{L}_2)^{\frac{3}{2}}. \tag{5}$$

Here the noninteger powers for the Lagrangian densities have been chosen appropriately to avoid Derrick's theorem. More models together with some explicit soliton solutions have been constructed, e.g., in [10, 11].

Among these models the AFZ model is special, because it has infinitely many symmetries and, as a consequence, infinitely many conservation laws [12, 13]. Further, infinitely many soliton solutions can be found by an explicit

integration for a special ansatz (separation of variables in toroidal coordinates), which realizes the concept of integrability in a rather explicit way. The other models do not have infinitely many symmetries, but, nevertheless, "integrable" subsectors with infinitely many conserved currents can be defined [14, 15]. The condition which defines these integrable subsectors is the complex eikonal equation

$$u^\mu u_\mu = 0 \tag{6}$$

where $u_\mu \equiv \partial_\mu u$. The infinitely many conserved currents J^G_μ (defined in Section 3) for these submodels are parametrized by an arbitrary, real function $G(u, \bar{u})$ and are, in fact, just the Noether currents for the area-preserving diffeomorphisms on target space [12, 16]. [Some more ("generalized") integrability conditions, which, however, depend on the Lagrangian, have been introduced in [17], [16].]

Here we want to demonstrate that there exists, instead of the complex eikonal equation, a weaker condition which again defines submodels with infinitely many conservation laws. Further, these integrable submodels can be defined for all Lagrangians for which the complex eikonal equation defines integrable submodels. Explicitly this condition reads

$$\bar{u}^2 u_\mu^2 - u^2 \bar{u}_\mu^2 = 0. \tag{7}$$

The infinitely many conserved currents J^G_μ for these submodels are as above, but with the additional restriction that now $G = G(u\bar{u})$. They are the Noether currents for an abelian subgroup of the group of area-preserving diffeomorphisms on target space.

The meaning of condition (7) becomes especially transparent when we re-express u in terms of its modulus and phase like

$$u = \exp(\Sigma + i\phi). \tag{8}$$

Then the complex eikonal equation is equivalent to the two real equations

$$\Sigma_\mu^2 = \phi_\mu^2 \tag{9}$$

and

$$\Sigma^\mu \phi_\mu = 0 \tag{10}$$

whereas the weaker condition (7) becomes Eq. (10) alone or, for time-independent u,

$$(\nabla\Sigma) \cdot (\nabla\phi) = 0. \tag{11}$$

The integrability condition (7) might be quite useful, for instance, in the case of the the Faddeev–Niemi model. For the Faddeev–Niemi model soliton solutions are only known numerically up to now [2, 18, 19, 20, 21]. No solutions which solve the complex eikonal equation, as well, are known and there are even arguments against the existence of such solutions [22]. On the other hand, it is perfectly possible that there exist solutions which solve the weaker integrability condition (7) and that this condition helps in the search for analytic solutions.

The condition (7) is in fact quite weak, i.e., quite easy to fulfill. For instance, many commonly used separation-of-variable ansätze, like the ansatz $u = \rho(r,\theta)e^{im\varphi}$ in spherical polar coordinates, or the ansatz $u = \rho(\eta)e^{i(m\varphi+n\xi)}$ in toroidal coordinates (both ρ are real), identically obey condition (7) due to the orthogonality of the corresponding basis vectors. On the other hand, for the eikonal equation these ansätze lead to a differential equation for the profile function ρ which only allows for very specific solutions, therefore providing a much stronger restriction, see, e.g., [23], [24]. In short, condition (7) applies to a rather large class of field configurations and, therefore, we believe that it will be useful for the study of non-linear field theories with a two-dimensional target space, like the Faddeev–Niemi or the Nicole model, or the other models mentioned above.

In Section 2 we discuss the algebra of generators of area-preserving diffeomorphisms and their abelian subalgebra on a two-dimensional manifold. Further we define the Noether charges corresponding to these generators. In Section 3 we show that condition (7) defines subsectors with infinitely many conservation laws for a very general class of Lagrangians (which cover all Lagrangians given above). Further we demonstrate that the corresponding conserved currents are indeed the Noether currents of the abelian area-preserving diffeomorphisms.

2 Abelian area-preserving diffeomorphisms

Here we describe area-preserving diffeomorphisms and an abelian subgroup contained within them for a two-dimensional manifold $\mathcal{M}$ which later on will be identified with the target space of the non-linear field theories which we want to study. Concretely, we choose real coordinates (ξ^1, ξ^2) or the complex

coordinate $u = \xi^1 + i\xi^2$ and allow for the class of metrics

$$ds^2 = g(a)[(d\xi^1)^2 + (d\xi^2)^2] = g(a) du d\bar{u} \tag{12}$$

where

$$a = (\xi^1)^2 + (\xi^2)^2 = u\bar{u} \tag{13}$$

and

$$du d\bar{u} \equiv \frac{1}{2}(du \otimes d\bar{u} + d\bar{u} \otimes du) \tag{14}$$

$$du \wedge d\bar{u} \equiv \frac{1}{2}(du \otimes d\bar{u} - d\bar{u} \otimes du). \tag{15}$$

The corresponding area two-form is

$$\Omega \equiv g(a) d\xi^1 \wedge d\xi^2 = \frac{g(a)}{2i} d\bar{u} \wedge du. \tag{16}$$

The choice of conformally flat metrics does not mean a restriction in two dimensions, because any metric on a two-dimensional manifold may be chosen conformally flat by an appropriate choice of coordinates. On the other hand, the functional dependence for the metric function $g = g(a)$ is a restriction, which is however sufficiently general for our purposes. In principle, one could skip this restriction, which would just complicate the subsequent discussion without adding substancial new structures (see the remark at the end of Section 3).

An area-preserving diffeomorphism is a transformation $u \to v(u, \bar{u})$ such that the area form (16) remains invariant (see also Refs. [12], [13], [15]),

$$\Omega \equiv \frac{1}{2i} g(u\bar{u}) d\bar{u} \wedge du = \frac{1}{2i} g(v\bar{v}) d\bar{v} \wedge dv. \tag{17}$$

For infinitesimal transformations $v = u + \epsilon$ it is easy to see that the condition of invariance of the area form leads to

$$\epsilon_u + \bar{\epsilon}_{\bar{u}} = -\frac{g'}{g}(\bar{u}\epsilon + u\bar{\epsilon}) \tag{18}$$

where $\epsilon_u \equiv \partial_u \epsilon$ and $g' \equiv \partial_a g(a)$. Defining

$$\epsilon = g^{-1}\delta\,, \quad \delta = F_{\bar{u}} \tag{19}$$

the above equation for ϵ simplifies to

$$\partial_u \partial_{\bar{u}} (F + \bar{F}) = 0. \tag{20}$$

The general solution to this equation is

$$F + \bar{F} = \zeta(u) + \bar{\zeta}(\bar{u}) \tag{21}$$

but for our purposes an imaginary F,

$$F + \bar{F} = 0, \tag{22}$$

serves as a general solution, because for any F which solves (21) there exists a $\tilde{F} = F - \zeta(u)$ which is imaginary and leads to the same $\delta = F_{\bar{u}} = \tilde{F}_{\bar{u}}$, i.e., to the same area-preserving diffeomorphism.

Introducing the real function G via $F = iG$, the area-preserving diffeomorphisms are therefore generated by the vector fields

$$v^G = ig^{-1}(G_{\bar{u}} \partial_u - G_u \partial_{\bar{u}}) \tag{23}$$

which obey the Lie algebra

$$[v^{G_1}, v^{G_2}] = v^{G_3}\,, \quad G_3 = ig^{-1}(G_{1,\bar{u}} G_{2,u} - G_{1,u} G_{2,\bar{u}}). \tag{24}$$

Now we want to find an abelian subalgebra of this Lie algebra of vector fields. It is easy to see that the commutator (24) vanishes if both $G_i, i = 1, 2$ are of the form

$$G = G(u\bar{u}). \tag{25}$$

In addition, this gives a maximal abelian subalgebra in the sense that if $G_1 = G_1(u\bar{u})$ then $G_3 = 0 \Leftrightarrow G_2 = G_2(u\bar{u})$. These issues may be seen especially easily by introducing the modulus and phase of u, $u = \sqrt{a} e^{i\phi}$. Then the vector field v^G for $G = G(a)$ is

$$v^G = H(a) \partial_\phi \quad , \qquad H(a) \equiv g^{-1} G' \tag{26}$$

and the above statements follow immediately. In short, the G of the form $G = G(u\bar{u})$ generate a maximal abelian subgroup of the group of area-preserving diffeomorphisms.

Due to the abelian nature of this subgroup it is trivial to integrate the infinitesimal transformations to reach finite ones. The result is that the transformations

$$u \to e^{i\Lambda(u\bar{u})}u \tag{27}$$

form a subgroup of abelian area-preserving diffeomorphisms, where $\Lambda = \Lambda(a)$ is an arbitrary function of its argument. In fact, these transformations leave invariant the two terms $g(a)$ and $d\bar{u} \wedge du$ separately.

Finally, let us describe how these transformations are implemented for field theories. For fields $u : \mathbb{R}^d \times \mathbb{R} \to \mathcal{M} : (\vec{x}, t) \to u(\vec{x}, t)$ the generators of area-preserving diffeomorphisms are given by Noether charges which are constructed with the help of the canonical momenta π, $\bar{\pi}$ of the fields u and $\bar{u}$. Concretely, they read

$$Q^G = i \int d^d\mathbf{x} g^{-1}(\bar{\pi}G_u - \pi G_{\bar{u}}) \tag{28}$$

and act on functions of $u, \bar{u}, \pi, \bar{\pi}$ via the Poisson bracket, where the fundamental Poisson bracket is (with $x^0 = y^0$)

$$\{u(\mathbf{x}), \pi(\mathbf{y})\} = \{\bar{u}(\mathbf{x}), \bar{\pi}(\mathbf{y})\} = \delta^d(\mathbf{x} - \mathbf{y}) \tag{29}$$

as usual. The generators Q^{G_i} close under the Poisson bracket, $\{Q^{G_1}, Q^{G_2}\} = Q^{G_3}$ where G_3 is as in (24). Specifically, for $G = G(a)$ they generate the abelian area-preserving diffeomorphisms, as above.

3 Integrable subsectors

In this section we want to show that for a wide class of Lagrangian densities integrable subsectors can be defined which have infinitely many conserved Noether currents which may be related to the abelian diffeomorphisms of the above section. The discussion in this section in some respect resembles the discussion in Ref. [16]. However, the integrability condition which we shall derive here has not been discussed in that reference. We introduce the class of Lagrangian densities

$$\mathcal{L}(u, \bar{u}, u_\mu, \bar{u}_\mu) = \mathcal{F}(a, b, c) \tag{30}$$

where

$$a = u\bar{u}\,, \quad b = u_\mu \bar{u}^\mu\,, \quad c = (u_\mu \bar{u}^\mu)^2 - u_\mu^2 \bar{u}_\nu^2 \tag{31}$$

and $\mathcal{F}$ is at this moment an arbitrary real function of its arguments. That is to say, we allow for Lagrangian densities which depend on the fields and on their first derivatives, are Lorentz invariant, real, and obey the phase symmetry $u \to e^{i\lambda}u$ for a constant $\lambda \in \mathbb{R}$. We could relax the last condition and allow for real Lagrangian densities which depend on u and $\bar{u}$ independently, but this would just complicate the subsequent discussion without adding anything substantial. Further, all models we want to cover fit into the general framework provided by the class of Lagrangian densities (30), therefore we restrict our discussion to this class.

The canonical four-momentum for this class of models is

$$\pi_\mu \equiv \mathcal{L}_{u^\mu} = \bar{u}^\mu \mathcal{F}_b + 2(u^\lambda \bar{u}_\lambda \bar{u}_\mu - \bar{u}_\lambda^2 u_\mu)\mathcal{F}_c \tag{32}$$

and the equation of motion reads

$$\partial^\mu \pi_\mu = \mathcal{L}_u = \bar{u}\mathcal{F}_a \tag{33}$$

together with its complex conjugate.

We introduce the infinitely many currents

$$J_\mu^G = if(a)(G_u \bar{\pi}_\mu - G_{\bar{u}}\pi_\mu) \tag{34}$$

where $f(a)$ is an arbitrary but fixed real function of its argument. Further, G is an arbitrary real function of u and $\bar{u}$, and $G_u \equiv \partial_u G$. Comparing with the Noether charge (28) it is tempting to identify $f = g^{-1}$ and J_μ^G with the Noether currents of area-preserving diffeomorphisms, and we will see in a moment that for a large subclass of Lagrangian densities this identification can be made, indeed.

In a first step, let us investigate which conditions make the divergence of the above current vanish, $\partial^\mu J_\mu^G = 0$. We find after a simple calculation

$$\begin{aligned} \partial^\mu J_\mu^G &= if\Big([(M'\bar{u}G_u + G_{uu})u_\mu^2 - (M'uG_{\bar{u}} + G_{\bar{u}\bar{u}})\bar{u}_\mu^2]\mathcal{F}_b \\ &\quad + (uG_u - \bar{u}G_{\bar{u}})[M'(b\mathcal{F}_b + 2c\mathcal{F}_c) + \mathcal{F}_a]\Big) \end{aligned} \tag{35}$$

where

$$M \equiv \ln f \tag{36}$$

and the prime denotes the derivative with respect to a.

The condition that the second term at the r.h.s. of Eq. (35) vanishes requires that either

$$uG_u - \bar{u}G_{\bar{u}} = 0 \tag{37}$$

or

$$M'(b\mathcal{F}_b + 2c\mathcal{F}_c) + \mathcal{F}_a = 0. \tag{38}$$

Assuming condition (37) we find the general solution

$$G(u, \bar{u}) = G(u\bar{u}) \equiv G(a) \tag{39}$$

which is exactly equal to the condition (25) which restricts the generators of area-preserving diffeomorphisms to the abelian subalgebra.

The condition that the first term at the r.h.s. of Eq. (35) vanishes requires that either

$$\mathcal{F}_b = 0 \tag{40}$$

or that

$$[(M'\bar{u}G_u + G_{uu})u_\mu^2 - (M'uG_{\bar{u}} + G_{\bar{u}\bar{u}})\bar{u}_\mu^2] = 0. \tag{41}$$

Condition (40) may, e.g., be satisfied by assuming $\mathcal{F}_b \equiv 0 \Rightarrow \mathcal{F} = \mathcal{F}(a, c)$. It follows that theories with Lagrangians $\mathcal{L} = \mathcal{F}(a, c)$ have infinitely many conserved currents (34), where G is restricted to (39). Of the models mentioned in the Introduction, only the AFZ model falls into this class. However, the AFZ model also obeys condition (38), therefore the restriction (39) is unnecessary and the J_μ^G are conserved for all G.

Alternatively we may make the first term at the r.h.s of Eq. (35) vanish by imposing Eq. (41). For an unrestricted G this leads to a condition on the field u,

$$u_\mu^2 = 0, \tag{42}$$

i.e., the complex eikonal equation, which, therefore, defines a submodel for which there exist infinitely many conserved currents provided that one of the two conditions (37) or (38) is imposed, in addition.

However, by envoking condition (39) we may re-express condition (41) like

$$(M'G' + G'')F_b[\bar{u}^2 u_\mu^2 - u^2 \bar{u}_\mu^2] \tag{43}$$

and, therefore, we find, instead of the complex eikonal equation, the weaker integrability condition

$$\bar{u}^2 u_\mu^2 - u^2 \bar{u}_\mu^2 = 0, \tag{44}$$

i.e., Eq. (7). Therefore, for *all* Lagrangians $\mathcal{L} = \mathcal{F}(a,b,c)$ condition (44) defines submodels which have infinitely many conserved currents (34), where G is restricted to (39), again. All models mentioned in the Introduction belong to this class.

Finally, we want to investigate what happens if we impose condition (38), either alternatively or in addition to condition (39) (we want to remark that condition (38) is fulfilled by all models mentioned in the Introduction). Equation (38) can be solved easily by the method of characteristics and has the general solution

$$\mathcal{F}(a,b,c) = \mathcal{F}(\frac{b}{f}, \frac{c}{f^2}). \tag{45}$$

This solution allows to interpret the Lagrangian in terms of the target space geometry and to identify the currents (34) with the Noether currents of the area-preserving diffeomorphisms of Section 2, as we want to demonstrate briefly. Indeed, trading the complex u field for two real target space coordinates ξ^α, $u \to (\xi^1, \xi^2)$, the expressions on which $\mathcal{F}$ may depend can be expressed as follows. The first term is

$$\frac{b}{f} = \frac{u_\mu \bar{u}^\mu}{f} = g_{\alpha\beta}(\xi)\partial^\mu \xi^\alpha \partial_\mu \xi^\beta \tag{46}$$

where $\alpha = 1, 2$ etc, and the target space metric $g_{\alpha\beta}$ is diagonal and conformally flat for the coordinate choice $\xi^1 = \mathrm{Re}\, u$, $\xi^2 = \mathrm{Im}\, u$, i.e.,

$$g_{\alpha\beta} = g(a)\delta_{\alpha\beta} \equiv f^{-1}\delta_{\alpha\beta}. \tag{47}$$

For the second term we get

$$\frac{c}{f^2} = \tilde{\epsilon}_{\alpha\beta}\tilde{\epsilon}_{\gamma\delta}\partial^\mu \xi^\alpha \partial_\mu \xi^\gamma \partial^\nu \xi^\beta \partial_\nu \xi^\delta \tag{48}$$

where

$$\tilde{\epsilon}_{\alpha\beta} = g\, \epsilon_{\alpha\beta} \quad , \quad g = f^{-1} = \det{}^{\frac{1}{2}}\,(g_{\gamma\delta}) \tag{49}$$

and $\epsilon_{\alpha\beta}$ is the usual antisymmetric symbol in two dimensions. We remark that the two terms are different in that the first one, b/f, depends on the target space metric, whereas the second one only depends on the determinant of the target space metric. For this class of Lagrangians the currents (34) are the Noether currents of area-preserving diffeomorphisms on target space, and

the condition $G = G(a)$ defines these Noether currents for the subgroup of abelian area-preserving diffeomorphisms defined in Section 2, as announced.

Remark: The abelian subalgebra spanned by generators of the form $G = G(u\bar{u})$ is by no way the only abelian subalgebra that exists for the algebra of vector fields v^G of Eq. (24). In fact, any subset of G of the form $G(u,\bar{u}) = \tilde{G}_i[h(u,\bar{u})]$ where h is an arbitrary but fixed function forms an abelian subalgebra, i.e. $[v^{\tilde{G}_1}, v^{\tilde{G}_2}] = 0$. This follows from the fact that for an area-preserving diffeomorphism the vector field $v^{\tilde{G}}$ has to be perpendicular to the (target space) gradient of h, i.e., it has to point into the direction $h =$ const. Indeed,

$$v^{\tilde{G}} h = i\tilde{G}'(h_{\bar{u}}\partial_u - h_u \partial_{\bar{u}})h = i\tilde{G}'(h_{\bar{u}}h_u - h_u h_{\bar{u}}) = 0. \tag{50}$$

However, these abelian subalgebras for $h \neq u\bar{u}$ do not play a special role in our discussion, i.e., they do not produce new integrability conditions. The reason why $h = u\bar{u}$ plays a special role lies in the fact that our metric function (Weyl factor) g depends on it, $g = g(u\bar{u})$. Had we chosen a different functional dependence $g = g[h(u,\bar{u})]$ for the metric function, then the corresponding generators $\tilde{G}_i[h(u,\bar{u})]$ of an abelian subalgebra would define a nontrivial new integrability condition. E.g. in the case $g = g(\xi^1) \equiv g(\frac{u+\bar{u}}{2})$ we find the integrability condition

$$u_\mu^2 - \bar{u}_\mu^2 = 0 \qquad \text{or} \qquad (\xi^1)_\mu (\xi^2)^\mu = 0. \tag{51}$$

A target space with a metric of the form $g = g(\xi^1)$, however, does not have the topology of the two-sphere (but rather the topology of $\mathbb{R}^2$ or of a cylinder). Therefore, the corresponding field theory does not have a nontrivial Hopf index and, consequently, does not give rise to knot solitons. In this sense it is, therefore, less interesting.

Acknowledgement:

This research was partly supported by MCyT(Spain) and FEDER (FPA2002-01161), Incentivos from Xunta de Galicia and the EC network "EUCLID". Further, CA acknowledges support from the Austrian START award project FWF-Y-137-TEC and from the FWF project P161 05 NO 5 of N.J. Mauser. AW gratefully acknowledges support from the Foundation for Polish Science FNP.

References

[1] L.D. Faddeev, in “40 Years in Mathematical Physics”, World Scientific, Singapore 1995.

[2] L.D. Faddeev, A.J. Niemi, Nature 387 (1997) 58; hep-th/9610193.

[3] E. Babaev, L.D. Faddeev, A. J. Niemi, Phys. Rev. B65 (2002) 100512.

[4] E. Babaev, Phys. Rev. Lett. 89 (2002) 067001.

[5] L.D. Faddeev, A. J. Niemi, Phys. Lett. B525 (2002) 195.

[6] L.D. Faddeev, A.J. Niemi, and U. Wiedner, Phys. Rev. D70 (2004) 114033.

[7] H. Aratyn, L.A. Ferreira, and A. Zimerman, Phys. Lett. B456 (1990) 162.

[8] H. Aratyn, L.A. Ferreira, and A. Zimerman, Phys. Rev. Lett. 83 (1999) 1723.

[9] D.A. Nicole, J. Phys. G4 (1978) 1363.

[10] A. Wereszczyński, Eur. Phys. J. C38: (2004) 261.

[11] A. Wereszczyński, Eur. Phys. J. C41 (2005) 265.

[12] O. Babelon and L.A. Ferreira, JHEP 0211 (2002) 020.

[13] L.A. Ferreira and A.V. Razumov, Lett. Math. Phys. 55 (2001) 143.

[14] O. Alvarez, L.A. Ferreira, and J. Sánchez-Guillén, Nucl. Phys. B529 (1998) 689.

[15] C. Adam and J. Sánchez-Guillén, JHEP 0501:004,2005.

[16] C. Adam and J. Sánchez-Guillén, Phys. Lett. B626 (2005) 235.

[17] A. Wereszczyński, Phys. Lett. B621 (2005) 201.

[18] J. Gladikowski and M. Hellmund, Phys. Rev. D56 (1997) 5194.

[19] R.A. Battye and P. Sutcliffe, Proc. Roy. Soc. Lond. A455 (1999) 4305.

[20] R.A. Battye and P. Sutcliffe, Phys. Rev. Lett. 81 (1998) 4798.

[21] J. Hietarinta, P. Salo, Phys. Rev. D62 (2000) 081701.

[22] Work in progress.

[23] C. Adam, J. Math. Phys. 45 (2004) 4017.

[24] A. Wereszczyński, Mod. Phys. Lett. A20 (2005) 1135.